SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2014

HIGHER SCHOOL CERTIFICATE ASSESSMENT TASK #3

Mathematics Extension 1

General Instruction

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Start each **NEW** question in a separate answer booklet.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question if full marks are to be awarded
- Answer in simplest exact form unless otherwise instructed.
- A table of standard integrals is provided
- There are 4 questions

Total Marks – 75

Question 1 – **20 Marks**

Question 2 – **20 Marks**

Question 3 – **15 Marks**

Question 4 – **20 Marks**

Attempt all questions

Examiner: *R. Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Question One (20 marks)

Marks

Use a SEPARATE answer booklet.

(a) Differentiate with respect to x, $\tan^{-1} 3x$. **2**

(b) Find **2**

$$\int x\sqrt{x^2+2}\,.\,dx$$

using the substitution $u = x^2 + 2$.

(c)

(i) Draw a neat sketch of $y = \sin^{-1} x$. **1**

(ii) State the domain and range. **2**

(iii) Using your sketch in Questions 1 (c) (i), shade the region bounded by the curve $y = \sin^{-1} x$, the x axis and the line $x = 1$. Using Simpson's Rule with 3 function values, find the approximation to this shaded area correct to 2 decimal places. **4**

(d) Given that **3**

$$\int_0^1 \frac{dx}{x^2+3} = A\pi$$

find the exact value of A.

(e) A particle moves on a line so that its distance x from the origin at time t is given by $x = t^3 - 9t^2 + 15t - 7$. Find when and where the particle first comes to rest. **3**

(f) Find the coordinates of the vertex and the focus and the equation of the directrix given the parabola $y = x^2 - 4x$. **3**

End of Question One

Question Two (20 marks)

Marks

Use a SEPARATE answer booklet.

(a) The rate at which a body cools in air is assumed to be proportional to the difference between its temperature T and the constant temperature S of the surrounding air. This can be expressed by the differential equation

$$\frac{dT}{dt} = k(T - S)$$

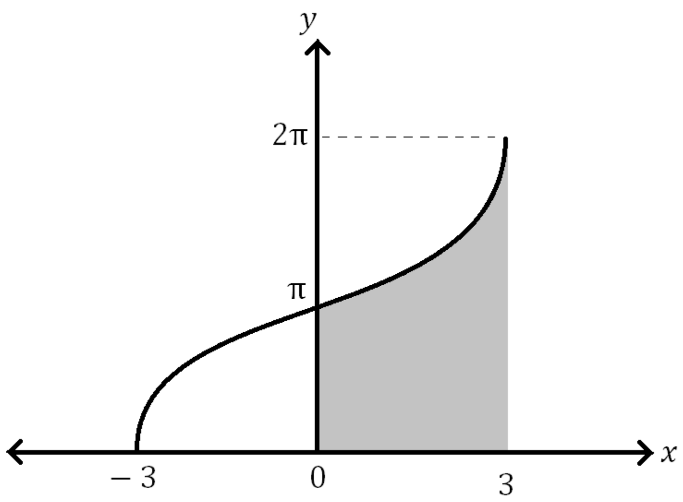
where t is the time in hours and k is a constant.

(i) Show that $T = S + Be^{kt}$, where B is a constant is a solution of the differential equation. **2**

(ii) A heated meal cools from 80°C to 40°C in 2 hours. The air temperature S in the room where the meal is placed is 20°C. Find the value of k correct to 4 decimal places. **2**

(iii) Find the temperature of the meal, correct to the nearest degree after 1 further hour has elapsed. **1**

(b) The graph below is the function $y = a\cos^{-1} bx$ where a and b are constants.



(i) Find the values of a and b. **2**

(ii) Hence find the exact value of the shaded area. **4**

(c) A particle moves along the x axis with acceleration $\frac{d^2x}{dt^2} = 70 + 12t - 12t^2$. **4**
If the particle is initially at rest at the origin, find its maximum displacement in the positive direction.

Question Two continues on next page

(d) A particle undergoes Simple Harmonic Motion about the origin O. Its displacement x cm from O at time t seconds is given by

$$x = 3\cos\left(2t + \frac{\pi}{3}\right)$$

(i) Express the acceleration as a function of the displacement. **2**

(ii) What is the amplitude of the motion? **1**

(iii) Find the value of x for which the speed is a maximum and determine this speed. **2**

End of Question Two

Question Three (15 marks)

Marks

Use a SEPARATE answer booklet.

(a) A spherical bubble is expanding so that its volume is increasing at the constant rate of $10mm^3$ per second. **3**
What is the rate of increase of the radius when the surface area is $500mm^2$.

(b)

(i) Differentiate with respect to x, **2**

$$x\sin^{-1}x + \sqrt{1-x^2}$$

(ii) Hence, evaluate **2**

$$\int_0^1 \sin^{-1}x\,.\,dx$$

leaving your answer in exact form.

(c) Use the substitution $u = x^3 + 3x - 2$ to evaluate **3**

$$\int_0^1 (x^2+1)\sqrt[3]{(x^3+3x-2)^5}\,.\,dx$$

(d) A particle moves on a line so that its distance from the origin O at time t is x.

(i) Given $\frac{d^2x}{dt^2} = 10x - 2x^3$ and $v = 0$ at $x = 1$, find v in terms of x. **3**

(ii) Is the motion Simple Harmonic? Explain. **2**

End of Question Three

Question Four (20 marks)

Marks

Use a SEPARATE answer booklet.

(a)

(i) Show that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ **2**

(ii) Hence or otherwise, find the positive value of x satisfying the equation **3**

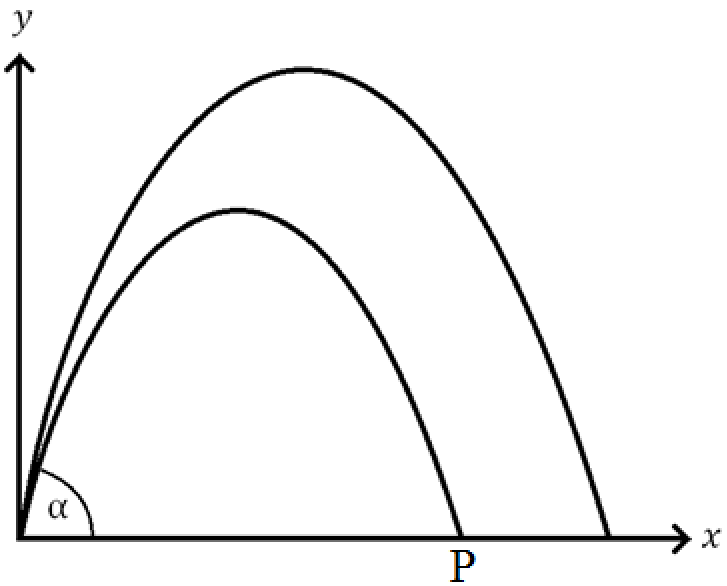
$$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$$

(b) A particle executes Simple Harmonic Motion with period T seconds and amplitude A cm. Find its maximum velocity in terms of A and T. **3**

(c) A cricket ball and a tennis ball are thrown simultaneously from the same point and in the same direction and with the same non-zero angle of projection (upward inclination to the horizontal), α but with different velocities U and V metres per second, where $U < V$. **5**

The slower tennis ball hits the ground first at a point P on the same level as the point of projection.

Show that, while the 2 balls are in flight, the line joining them has an inclination to the horizontal which is independent of time.



Question Four continues on next page

(d) Tangents are drawn from the external point $P(x_0, y_0)$ to the parabola $x^2 = 4y$. These tangents touch the parabola at Q and R respectively.

(i) Prove that the midpoint T of QR is **5**

$$\left(x_0, \frac{1}{2}x_0^2 - y_0\right)$$

(ii) If P moves on the line $x - y = 1$, find the equation of the locus of T. **2**

End of Question Four
End of Exam

SBHS 2014 Extension 1 Assessment #3 Solutions

Question One

(a.) $y = \tan^{-1} 3x$

$y' = \boxed{\dfrac{3}{1+9x^2}}$

(b.) Let $u = x^2 + 2$

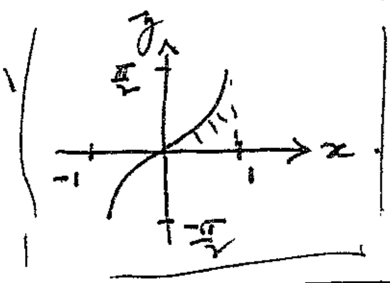
$du = 2x\,dx$.

$\therefore I = \frac{1}{2}\int \sqrt{u}\,.\,du$

$= \frac{1}{3} u^{3/2} + c$

$= \boxed{\frac{1}{3}(x^2+2)^{3/2} + c}$

(c) (i). (1)



(ii) $D: \quad |x| \le 1.$

$R: \quad |\sin^{-1} x| \le \frac{\pi}{2}.$

(iii) $A \doteqdot \dfrac{\frac{1}{2}}{3}\left[f(0) + f(1) + 4f(\tfrac{1}{2})\right]$

$= \frac{1}{6}\left[0 + \frac{\pi}{2} + 4 \times \frac{\pi}{6}\right]$

$= \frac{1}{6}\left(\frac{\pi}{2} + \frac{2\pi}{3}\right)$

$= \dfrac{7\pi}{36}.$

$\doteqdot \boxed{0\cdot 61 \ u^2}$

(d). $\int_0^1 \frac{dx}{x^2+3} = A\pi$.

$LHS = \frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{x}{\sqrt{3}}\right]_0^1$

$= \frac{1}{\sqrt{3}} \tan^{-1}\frac{1}{\sqrt{3}}$.

$= \frac{1}{\sqrt{3}} \cdot \frac{\pi}{6}$.

$\therefore \frac{\pi}{6\sqrt{3}} = A\pi$.

$\therefore A = \frac{1}{6\sqrt{3}}$

$= \boxed{\frac{\sqrt{3}}{18}}$

(e) $x = t^3 - 9t^2 + 15t - 7$.

$\dot{x} = 3t^2 - 18t + 15$

$= 3(t^2 - 6t + 5)$

$= 3(t-1)(t-5)$

$\therefore$ at rest when $\dot{x} = 0$

$t = 1, 5$.

$\therefore$ first time is after 1 units of time

at $x = 0$.

(f) $y = x^2 - 4x$

$\therefore (x-2)^2 = y + 4$

$(x-2)^2 = 4 \times \frac{1}{4}(y+4)$

$\therefore$ Vertex is $(2, -4)$

Focus. $(2, -3\frac{3}{4})$

DIRECTRIX. $y = -4\frac{1}{4}$.

SHS Ext1 Task 3 2014

Question 2

(a) (i) $T = S + Be^{kt} \quad \therefore Be^{kt} = T - S$

$$\begin{aligned}\frac{dT}{dt} &= kBe^{kt}\\ &= k(T - S)\end{aligned}$$

$\therefore$ It is a solution to the DE

(ii) $S = 20$

Now $80 = 20 + Be^{k0}$ when $t = 0$

$\therefore B = 60$

When $t = 2$, $T = 40$

$$\begin{aligned}40 &= 20 + 60e^{2k}\\ 20 &= 60e^{2k}\\ e^{2k} &= \tfrac{1}{3}\\ 2k &= \ln\tfrac{1}{3}\\ k &= \tfrac{1}{2}\ln\tfrac{1}{3}\\ &\approx -0.5493\end{aligned}$$

(iii) When $t = 3$

$$\begin{aligned}T &= 20 + 60e^{3k}\\ &\approx 31\cdot 55^{0}\end{aligned}$$

(b) (i) $y = a\cos^{-1} bx$

When $x = 0,\ y = \pi$

$\pi = a\cos^{-1} b \times 0$

$\pi = a\frac{\pi}{2}$

Thus $a = 2$

When $x = 3,\ y = 2\pi$

$2\pi = 2\cos^{-1} 3b$

$\cos\pi = 3b$

Thus $b = -\frac{1}{3}$

$$\therefore y = 2\cos^{-1}\left(-\frac{x}{3}\right)$$

(ii) $\frac{y}{2} = \cos^{-1}\left(-\frac{x}{3}\right)$

$$-\frac{x}{3} = \cos\left(\frac{y}{2}\right)$$
$$x = -3\cos\left(\frac{y}{2}\right)$$

Area consists of rectangle plus area between line $x = 3$ and curve.

$$A = 3\pi + \int_{\pi}^{2\pi}\left(3 - \left(-3\cos\left(\frac{y}{2}\right)\right)\right)dy$$
$$= 3\pi + \left[3y + 3\times 2\sin\left(\frac{y}{2}\right)\right]_{\pi}^{2\pi}$$
$$= 3\pi + [6\pi + 6\sin\pi] - \left[3\pi + 6\sin\left(\frac{\pi}{2}\right)\right]$$
$$= 6\pi - 6 \text{ sq units}$$

(c) $$\frac{d^2y}{dx^2} = 70 + 12t - 12t^2$$

When $t = 0,\ x = 0,\ v = 0$.

$$\frac{dy}{dx} = 70t + 6t^2 - 4t^3 + C, \quad C = 0$$
$$\therefore v = 70t + 6t^2 - 4t^3$$

Hence $x = 35t^2 + 2t^3 - t^4 + D, \quad D = 0$

$$\therefore x = 35t^2 + 2t^3 - t^4$$

Max displacement when $v = 0$.

$$0 = 70t + 6t^2 - 4t^3$$
$$0 = 2t\left(35 + 3t - 2t^2\right)$$
$$t = 0, \frac{3 \pm 17}{4}$$
$$t = -\frac{7}{2},\ 0,\ 5$$

Clearly $t > 0$, so max x is when $t = 5$.

$$x_{\max} = 35(5)^2 + 2(5)^3 - (5)^4 = 500$$

(d) $x = 3\cos\left(2t + \tfrac{\pi}{2}\right)$

(i) $\dot{x} = -6\sin\left(2t + \tfrac{\pi}{2}\right)$

$\ddot{x} = -12\cos\left(2t + \tfrac{\pi}{2}\right)$

$\therefore \ddot{x} = -4x$

(ii) $a = 3$

(iii) Max speed when $\ddot{x} = 0$

$0 = -12\cos\left(2t + \tfrac{\pi}{2}\right)$

$\therefore \cos\left(2t + \tfrac{\pi}{2}\right) = 0$

$$2t + \frac{\pi}{2} = \frac{\pi}{2}$$

$\therefore t = 0$

At this time

$$\dot{x} = -6\sin\left(\frac{\pi}{2}\right)$$

$$= -6$$

Hence max speed = 6 cm/s.

Q.3. Extension 1 – 15 Marks

(a) $\frac{dV}{dt} = 10 mm^3/s$ Find $\frac{dr}{dt}$ when $S = 500 mm^2$

Now $V = \frac{4}{3}\pi r^3$ ✓

$\Rightarrow S = \frac{dV}{dr} = 4\pi r^2$ ✓

Then $\frac{dV}{dr} = \frac{dV}{dt}\frac{dt}{dr}$

$500 = 10 \times \frac{dt}{dr}$

$\frac{dt}{dr} = 50$ ✓ (3)

$\Rightarrow \frac{dr}{dt} = \frac{1}{50}$ mm/s.

(b) (i) $y = x\sin^{-1}x + \sqrt{1-x^2}$

$y' = x.\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x.1 + \frac{1}{2}(1-x^2)^{-1/2}(-2x)$ ✓

$y' = \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$ ✓ (2)

$\Rightarrow y' = \sin^{-1}x.$

(ii) $\therefore \int_0^1 \sin^{-1}x\,dx = \left[x.\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$ ✓

$= (\sin^{-1}1 + 0) - (0 + 1)$

$= \sin^{-1}1 - 1$

$= \frac{\pi}{2} - 1.$ ✓ (2)

3(c) $\int_0^1 (x^2+1)\,.\sqrt[3]{(x^3+3x-2)^5}\,dx$ Let $u = x^3+3x-2$, $du = 3(x^2+1)dx$

$I = \int (x^2+1)\, u^{5/3} \frac{du}{3(x^2+1)}$ ✓

$= \frac{1}{3}\int u^{5/3}\,du$

$= \frac{1}{3} u^{8/3} \times \frac{3}{8}$ ✓

$= \frac{1}{8} u^{8/3}$ (3)

Then Definite Int $= \frac{1}{8}\left[(x^3+3x-2)^{8/3}\right]_0^1$

$= \frac{1}{8}\left(2^{8/3} - (-2)^{8/3}\right)$

$= 0$ ✓

(d)(i) $\ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx} = 10x - 2x^3$ ✓

$\frac{1}{2}v^2 = 5x^2 - \frac{x^4}{2} + c$

When $v=0, x=1 \Rightarrow 0 = 5 - \frac{1}{2} + c$ ✓

$\Rightarrow c = -4\frac{1}{2}$ (3)

$\therefore \frac{1}{2}v^2 = 5x^2 - \frac{x^4}{2} - 4\frac{1}{2}$

$v^2 = 10x^2 - x^4 - 9$ ✓

(ii) Since $\ddot{x} = 10x - 2x^3$

$\neq -n^2 x$ ✓

and $\neq -n^2(x-a)$ where $n \in Z$

$\Rightarrow$ NOT SHM. ✓ (2)

Q4

(a) (i) RTP $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\tan(\tan^{-1}x + \tan^{-1}y)$

$= \frac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\tan(\tan^{-1}y)}$

$= \frac{x+y}{1-xy}$

$\therefore \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ [2]

(ii) $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

$\therefore \tan^{-1}\left(\frac{2x+3x}{1-2x.3x}\right) = \frac{\pi}{4}$

$\therefore \frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$

$\therefore 5x = 1-6x^2$

$\therefore 6x^2+5x-1 = 0$

$\therefore (6x-1)(x+1) = 0$

$\therefore x = \frac{1}{6}$ OR $x = -1.$

[3]

As $x > 0$, $x = \frac{1}{6}$.

(b) Particle executes SHM

$\therefore v^2 = n^2(a^2 - x^2)$

As Period $= T = \frac{2\pi}{n}$

$n = \frac{2\pi}{T}$

$\therefore v^2 = \left(\frac{2\pi}{T}\right)^2 (A^2 - x^2)$

Max velocity occurs when $x = 0$

$\therefore v_{max} = \frac{2\pi A}{T}$

3

(c) Tennis ball:

$\ddot{x} = 0$

$\dot{x} = U\cos\alpha$

$x = U\cos\alpha\, t$

$\ddot{y} = -g$

$\dot{y} = -gt + U\sin\alpha$

$y = -\frac{1}{2}gt^2 + U\sin\alpha\, t$

Cricket ball

$\ddot{x} = 0$

$\dot{x} = V\cos\alpha$

$x = V\cos\alpha\, t$

$\ddot{y} = -g$

$\dot{y} = -gt + V\sin\alpha$

$y = -\frac{1}{2}gt^2 + V\sin\alpha\, t$

Gradient between balls

$$= \frac{-\frac{1}{2}gt^2 + V\sin\alpha\, t - \left(-\frac{1}{2}gt^2 + U\sin\alpha\, t\right)}{V\cos\alpha\, t - U\cos\alpha\, t}$$

$$= \frac{V\sin\alpha\, t - U\sin\alpha\, t}{V\cos\alpha\, t - U\cos\alpha\, t}$$

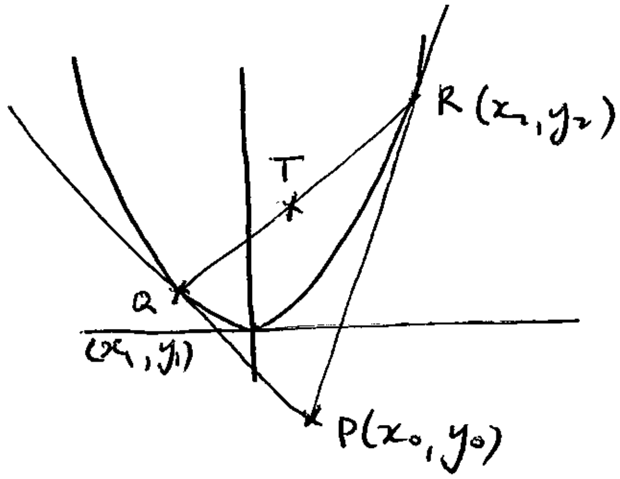
$$= \frac{(V-U)\sin\alpha\, t}{(V-U)\cos\alpha\, t}$$

$= \tan\alpha$ which is independent of t.

$\therefore$ Inclination to horizontal is independent of t

5

(d) (i)



QR is $xx_0 = 2(y + y_0)$

For Q and R $\quad xx_0 = 2\left(\frac{x^2}{4} + y_0\right)$

$\therefore xx_0 = \frac{x^2}{2} + 2y_0$

$\therefore 2xx_0 = x^2 + 4y_0$

$\therefore x^2 - 2xx_0 + 4y_0 = 0$

Sum of roots $\quad x_1 + x_2 = 2x_0$

$\therefore \frac{x_1 + x_2}{2} = x_0$

$\therefore x_T = x_0$

As T lies on QR

$x_T x_0 = 2(y_T + y_0)$

$\therefore x_0 x_0 = 2(y_T + y_0)$

$\therefore \frac{x_0^2}{2} = y_T + y_0$

$\therefore y_T = \frac{x_0^2}{2} - y_0$

$\therefore$ T is $\left(x_0, \frac{x_0^2}{2} - y_0\right)$

5

(ii) As P moves on $x - y = 1$

$$x_0 - y_0 = 1$$

$$\therefore y_0 = x_0 - 1$$

$$\therefore y_T = \frac{x_0^2}{2} - y_0$$

$$\therefore y_T = \frac{x_T^2}{2} - (x_0 - 1)$$

$$= \frac{x_T^2}{2} - x_T + 1.$$

$\therefore$ Locus of T is $y = \frac{x^2}{2} - x + 1$.

2